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On the discrete normalized gradient flow for computing ground states of rotating Bose–Einstein condensates: energy dissipation and global convergence

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The discrete normalized gradient flow with semi-implicit time discretization (GFSI), introduced in [W. Bao and Q. Du, SIAM J. Sci. Comput., 25 (2004), pp. 1674–1697], has become one of the most widely used numerical method for computing the ground states of Bose–Einstein condensates (BECs). In spite of strong numerical evidence of its energy dissipation and global convergence a rigorous proof of these two properties for the GFSI remains an open challenge. In this work we comprehensively study these two properties of the GFSI in the context of computing ground states of BECs with and without angular momentum rotation. By reformulating the GFSI into an equivalent form in terms of the normalized numerical solution we rigorously establish the energy dissipation property of the algorithm. Building on this result we further establish its global convergence to a stationary state. To the best of our knowledge this study represents the first rigorous derivation of both energy dissipation and global convergence for the GFSI.

Keywords: rotating Bose–Einstein condensates; ground states; normalized gradient flow; energy dissipation; global convergence.

1. Introduction

Since their experimental realization in 1995 (Anderson *et al.*, 1995; Davis *et al.*, 1995) Bose–Einstein condensates (BECs) have garnered significant interest of researchers across various disciplines. In BECs ultra-cold atoms are cooled to temperatures near absolute zero, causing them to occupy the same quantum state. This results in a coherent state of matter with remarkable quantum properties, offering profound insights into the macroscopic quantum world. Over the past two decades research on BECs has advanced significantly in areas such as atomic, molecular, optical and condensed matter physics. Particular emphasis has been placed on rotating BECs, where induced quantum vortices (Abo-Shaeer *et al.*, 2001; Fetter & Svidzinsky, 2001; Bretin *et al.*, 2004) play a crucial role. The presence of vortices provides evidence of the superfluid nature of BECs and has potential applications in various fields, including astrophysics, atomic physics, optics and superfluid dynamics (Klaers *et al.*, 2010).

At temperatures below the critical temperature T_c, the macroscopic behaviour of a rotating BEC can be accurately described by a complex-valued wave function $\psi := \psi(\mathbf{x}, t)$, whose evolution is governed by the dimensionless Gross–Pitaevskii equation (GPE) incorporating the angular momentum rotation term (Bao & Cai, 2013):

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V(\mathbf{x})\psi + \beta|\psi|^2\psi - \omega L_z\psi, \qquad \mathbf{x}\in\mathbb{R}^d, \quad t>0. \tag{1.1}$$

Here, t is time, $\mathbf{x}\in\mathbb{R}^d$ ($d=2,3$) is the spatial coordinate with $\mathbf{x}=(x,y)^T$ in two dimensions and $\mathbf{x}=(x,y,z)^T$ in three dimensions, L_z is the angular momentum operator defined as

$$L_z := -i(x\partial_y - y\partial_x)$$

and $\omega\geq 0$ denoting the rotation frequency. In the nonrotating case (i.e., $\omega=0$), without the angular momentum rotation term, we can also consider (1.1) in the one-dimensional setting with $d=1$ and $\mathbf{x}=x$. Moreover, $V=V(\mathbf{x})$ is a real-valued external potential and $\beta\in\mathbb{R}$ characterizes the two-body interaction in the condensate ($\beta>0$ for the defocusing/repulsive case and $\beta<0$ for the focusing/attractive case). There are two important conserved quantities of the time-dependent GPE (1.1): the mass (or normalization)

$$N(\psi(\cdot,t)) := \|\psi(\cdot,t)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}|\psi(\cdot,t)|^2\,\mathrm{d}\mathbf{x} \equiv N(\psi(\cdot,0)) = 1, \qquad t\geq 0, \tag{1.2}$$

and the total energy

$$\begin{aligned} E(\psi(\cdot,t)) &:= \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla\psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4 - \omega\overline{\psi}L_z\psi\right)\mathrm{d}\mathbf{x} \\ &\equiv E(\psi(\cdot,0)), \qquad t\geq 0, \end{aligned} \tag{1.3}$$

with $\overline{\psi}$ denoting the complex conjugate of ψ.

One important object in studies of BECs is the ground state, which is defined as a macroscopic wave function $\phi_g(\mathbf{x})$ that minimizes the energy functional E (1.3) under the mass constraint (1.2) (Bao & Cai, 2013), i.e.,

$$\phi_g(\mathbf{x}) := \arg\min_{\phi\in\mathscr{S}} E(\phi), \tag{1.4}$$

where $\mathscr{S} := \{\phi\in L^2(\mathbb{R}^d) \mid N(\phi)=1, E(\phi)<\infty\}$. The Euler–Lagrange equation associated with the above problem reads as

$$\mu\phi = -\frac{1}{2}\Delta\phi + V(\mathbf{x})\phi + \beta|\phi|^2\phi - \omega L_z\phi, \qquad \text{with} \quad \phi\in\mathscr{S}, \qquad \mathbf{x}\in\mathbb{R}^d,$$

which is an L^2-normalized nonlinear eigenvalue problem for (μ,ϕ). The corresponding eigenvalue μ can be computed through the eigenfunction ϕ as

$$\mu(\phi) = \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla\phi|^2 + V(\mathbf{x})|\phi|^2 + \beta|\phi|^4 - \omega\overline{\phi}L_z\phi\right)\mathrm{d}\mathbf{x} = E(\phi) + \frac{\beta}{2}\int_{\mathbb{R}^d}|\phi|^4\,\mathrm{d}\mathbf{x}. \tag{1.5}$$

Along the analytical side the existence and uniqueness of ground states as well as their various properties are extensively studied (see, e.g., Bao & Cai (2013)).

To compute the ground states of BEC a discrete normalized gradient flow (DNGF) was introduced in Bao & Du (2004), where a normalization (projection) step is imposed following each step of gradient descent. To further discretize the continuous-time gradient flow in the DNGF a semi-implicit time discretization is proposed, which leads to the discrete normalized gradient flow with semi-implicit time discretization (GFSI) recalled in (2.12)–(2.13). Due to the GFSI's simplicity and robust numerical performance it has become the most widely used numerical method for computing the ground states of BECs since then. However, although substantial numerical results have been reported to support the energy dissipating and global convergence of the GFSI, rigorous proof of these two properties remains an open challenge.

There have been several attempts in the literature intending to address these problems and to provide a rigorous theoretical foundation of the GFSI, mainly for the nonrotating case, i.e., $\omega = 0$. First, in the original article (Bao & Du, 2004), for the linear problem (i.e., $\beta = 0$), the energy dissipation of the GFSI is proved and for the nonlinear case with $\beta > 0$, a modified energy decay of the GFSI is presented. Additionally, the GFSI is also interpreted as a certain discretization of a continuous normalized gradient flow (CNGF) (see (2.11)), which is energy-dissipating and mass-conserving. Recently, for the case $\omega = 0$, the local convergence of the GFSI was obtained by Faou and Jézéquel (Faou & Jézéquel, 2017) for the focusing case in one dimension (i.e., $\beta < 0$), and by Henning (Henning, 2023), for the defocusing case (i.e., $\beta > 0$) in one dimension, two dimension and three dimension. However, the energy diminishing and the global convergence of the GFSI remain elusive. In addition to these attempts many variants of the GFSI and optimization-based methods, such as a large class of Sobolev gradient flows, were proposed and analysed (Chiofalo *et al.*, 2000; García-Ripoll & Pérez-García, 2001; Bao *et al.*, 2006; Bao & Wang, 2007; Cancés *et al.*, 2010; Danaila & Kazemi, 2010; Kazemi & Eckart, 2010; Antoine & Duboscq, 2014; Jarlebring *et al.*, 2014; Antoine *et al.*, 2017; Danaila & Protas, 2017; Wu *et al.*, 2017; Zhuang & Shen, 2019; Henning & Peterseim, 2020; Altmann *et al.*, 2021; Liu & Cai, 2021; Zhang, 2022; Chen *et al.*, 2023; Henning, 2023; Chen *et al.*, 2024; Ai *et al.*, 2026). Very recently, for some Sobolev gradient flows, the energy dissipation as well as global and local convergence are proved at both time-continuous and discrete levels (see Henning & Peterseim (2020); Henning (2023); Chen *et al.* (2024) and Henning & Yadav (2025)). However, these techniques cannot be applied to the original GFSI. Hence, even for the nonrotating case, the theoretical analysis of the GFSI remains largely open, and no rigorous proof of the dissipation of the original energy and the global convergence is available.

In this work we aim to close this gap and provide a comprehensive analysis of the GFSI in the context of computing the ground states of (rotating) BECs. Notably, we prove, for the first time, the energy dissipating and global convergence of the GFSI for both the focusing $\beta > 0$ and defocusing $\beta < 0$ cases with and without rotation. The key ingredient of the proof is to reformulate the GFSI in terms of the normalized numerical solution, which is reminiscent of the CNGF. Based on this new formulation we directly estimate the energy dissipation between two adjacent steps instead of using the nonnormalized numerical solution as an intermediary, which is used in Bao & Du (2004) for the GFSI and in, e.g., Henning & Peterseim (2020); Chen *et al.* (2024) and Henning & Yadav (2025), for the Sobolev gradient flows.

The rest of the paper is organized as follows: In Section 2 we introduce some necessary preliminaries, review the GFSI and present our main result. Section 3 is devoted to the proof of the main result, including the energy dissipation and global convergence of the GFSI, based on a more elaborate equivalent form. Section 4 is devoted to numerical tests to validate the theoretical findings. Finally, some conclusions are drawn in Section 5.

2. The algorithm and main results

In this section we introduce some basic notations, review mathematical formulation of the GFSI and present the main results.

In practical computations, the whole space $\mathbb{R}^d$ is always truncated into a bounded domain Ω with either homogeneous Dirichlet or periodic boundary conditions, which can be justified by the exponential decay of stationary states in the far field due to the trapping potential $V(\mathbf{x})$ (cf. Bao & Cai (2013)). To simplify the presentation we only consider the case of homogeneous Dirichlet boundary conditions and assume that the boundary is Lipschitz continuous. The analysis can be directly generalized to periodic boundary conditions and the main results remain unchanged.

2.1 *Notation*

On the bounded domain Ω we adopt standard notations for the Lebesgue spaces $L^p(\Omega) = L^p(\Omega;\mathbb{C})$ and the Sobolev space $H_0^1(\Omega) = H_0^1(\Omega;\mathbb{C})$ as well as the corresponding norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H_0^1}$. Here, we drop the Ω dependence in the norms to simplify the notations. Now, the energy functional E is well defined on $H_0^1(\Omega)$ as

$$E(\phi) := \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + V(\mathbf{x})|\phi|^2 + \frac{\beta}{2}|\phi|^4 - \omega\overline{\phi}L_z\phi\right) \mathrm{d}\mathbf{x}, \tag{2.1}$$

and the constraint set $\mathscr{S} \subset H_0^1(\Omega)$ can be characterized as

$$\mathscr{S} := \left\{\phi \in H_0^1(\Omega) \mid \|\phi\|_{L^2} = 1\right\}. \tag{2.2}$$

Under (2.1) and (2.2) the ground state problem can be formulated as

$$\phi_g := \arg\min_{\phi\in\mathscr{S}} E(\phi). \tag{2.3}$$

To ensure the existence of the ground state ϕ_g we make the following assumptions on the trapping potential V, rotation frequency ω and β: Assume that $V \in L^\infty(\Omega)$ satisfies $V(\mathbf{x}) \geq 0$ for $\mathbf{x} \in \Omega$ and, when $d \geq 2$,

$$V(\mathbf{x}) \geq (1+K)\omega^2(x^2+y^2)/2, \quad \mathbf{x} \in \Omega, \quad \text{for some constant } K > 0, \tag{2.4}$$

and one of the following holds:

(i) $d = 3$, $\beta \geq 0$;

(ii) $d = 2$, $\beta > -C_b$, where

$$C_b = C_b(\Omega) := \inf_{0\neq\phi\in H_0^1(\Omega)} \frac{\|\nabla\phi\|_{L^2}^2\|\phi\|_{L^2}^2}{\|\phi\|_{L^4}^4}; \tag{2.5}$$

(iii) $d = 1$, $\beta \in \mathbb{R}$.

These conditions are sufficient for the existence of ground states defined in (2.3) (see (Bao & Cai, 2013, theorems 2.1 and 5.1)) and will be assumed in the rest of this paper. In particular, when we refer to $\beta < 0$, it is implicitly assumed that either $d = 1$, $\beta < 0$ or $d = 2$ and $-C_b < \beta < 0$.

Since the energy is real-valued while the ground state solution ϕ_g might be complex-valued, we shall work within real linear spaces consisting of complex-valued functions (cf. Cazenave (2003) and Altmann *et al.* (2021)). To this end, we equip the Hilbert spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ with the following real inner products:

$$(v, w)_{L^2} := \operatorname{Re} \int_\Omega v\overline{w}\,\mathrm{d}\mathbf{x} \quad \text{and} \quad (v, w)_{H_0^1} := \operatorname{Re} \int_\Omega \nabla v \cdot \nabla \overline{w}\,\mathrm{d}\mathbf{x}.$$

The corresponding (real) dual space is denoted by $H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)^*$ with canonical duality pairing $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$. We define a Hamiltonian operator $H_\phi : H_0^1(\Omega) \to H^{-1}(\Omega)$ for $\phi \in H_0^1(\Omega)$ as

$$H_\phi := -\frac{1}{2}\Delta + V - \omega L_z + \beta|\phi|^2, \tag{2.6}$$

and introduce a symmetric bilinear form $\mathscr{A}_\phi^a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ associated with the Hamiltonian operator H_ϕ as follows: for $\phi \in H_0^1(\Omega)$ and $a > 0$

$$\begin{aligned}\mathscr{A}_\phi^a(u, v) &:= \langle (I + aH_\phi)u, v\rangle \\ &= \operatorname{Re}\int_\Omega u\overline{v}\,\mathrm{d}\mathbf{x} + a\operatorname{Re}\int_\Omega \left(\frac{1}{2}\nabla u \cdot \nabla\overline{v} + V(\mathbf{x})u\overline{v} + \beta|\phi|^2 u\overline{v} - \omega\overline{v}L_z u\right)\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.7}$$

For the Hamiltonian operator H_ϕ and the bilinear form $\mathscr{A}_\phi^a(\cdot,\cdot)$, we have the following estimates. Moreover, hereinafter, we denote by C a generic constant that may depend on Ω, d, K and $\|V\|_{L^\infty}$, but is independent of β. This particularly includes the constants from Sobolev inequalities.

Lemma 1. Given $\phi \in \mathscr{S}$ we have the following:

(i) For any u, $v \in H_0^1(\Omega)$ we have

$$\langle H_0 u, u\rangle \geq C_0\|u\|_{H_0^1}^2, \tag{2.8}$$

$$\left|\langle H_\phi u, v\rangle\right| \leq C_\phi\|u\|_{H_0^1}\|v\|_{H_0^1}, \tag{2.9}$$

where $H_0 = -\frac{1}{2}\Delta + V(\mathbf{x}) - \omega L_z$, $C_0 := K/\left(2(K+1)\right)$ and $C_\phi = C + C|\beta|\|\phi\|_{H_0^1}^{\frac{d}{2}}$.

(ii) The symmetric bilinear form $\mathscr{A}_\phi^a(\cdot,\cdot)$ is continuous for any $a > 0$ with

$$|\mathscr{A}_\phi^a(u, v)| \leq (1 + aC_\phi)\|u\|_{H_0^1}\|v\|_{H_0^1}, \qquad u, v \in H_0^1(\Omega).$$

(iii) The symmetric bilinear form $\mathscr{A}_\phi^a(\cdot,\cdot)$ is coercive for $0 < a \le a_\phi$ with

$$\mathscr{A}_\phi^a(u,u) \ge \frac{aC_0}{2}\|u\|_{H_0^1}^2, \qquad u \in H_0^1(\Omega),$$

where $a_\phi = +\infty$ when $\beta \ge 0$ and $a_\phi = C/(\beta^2\|\phi\|_{H_0^1}^d)^{\frac{2}{4-d}}$ when $\beta < 0$.

Proof. The proof is standard and can be found in the appendix. □

We remark here that when writing $0 < a \le a_\phi$ the case $a = +\infty$ is excluded.

REMARK 1. By Lemma 1 (ii) and (iii) and the Lax–Milgram theorem, when $0 < a \le a_\phi$, for any $w \in H^{-1}(\Omega)$, there exists a unique $u \in H_0^1(\Omega)$ such that

$$\mathscr{A}_\phi^a(u,v) = \big\langle (I + aH_\phi)u, v\big\rangle = \langle w, v\rangle \qquad \text{for all } v \in H_0^1(\Omega), \tag{2.10}$$

and we shall denote this u by $(I + aH_\phi)^{-1}w$ in the subsequent discussions. In fact, $u = (I + aH_\phi)^{-1}w \in H_0^1(\Omega)$ is the unique weak solution to the equation $(I + aH_\phi)u = w$ under the homogeneous Dirichlet boundary condition.

2.2 *The GFSI algorithm and main results*

In this subsection we review the GFSI algorithm proposed in Bao & Du (2004), which is obtained by applying the time-splitting method and a semi-implicit time discretization to the following CNGF:

$$\begin{cases} \partial_t\phi(\mathbf{x},t) = -H_\phi\phi(\mathbf{x},t) + \mu_\phi(t)\phi(\mathbf{x},t), & \mathbf{x} \in \Omega, \quad t > 0, \\ \phi(\mathbf{x},t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t > 0, \\ \phi(\mathbf{x},0) = \phi_0(\mathbf{x}) \in \mathscr{S}, \quad \mathbf{x} \in \Omega, \end{cases} \tag{2.11}$$

where $\mu_\phi(t) = \mu(\phi(\cdot,t))$. As mentioned before the solution ϕ to the CNGF (2.11) is mass-preserving and energy dissipating (Bao & Du, 2004).

We choose a time step size $\tau > 0$ and denote the time steps as $t_n = n\tau$ for $n = 0, 1, \ldots$. Let $\phi^n(\cdot)$ be the numerical approximation to $\phi(\cdot, t_n)$. Then, the GFSI reads

$$\frac{\widetilde{\phi}^{n+1} - \phi^n}{\tau} = -H_{\phi^n}\widetilde{\phi}^{n+1}, \quad \widetilde{\phi}^{n+1} = 0 \text{ on } \partial\Omega, \quad n \ge 0, \tag{2.12}$$

$$\phi^{n+1} = \widetilde{\phi}^{n+1}/\|\widetilde{\phi}^{n+1}\|_{L^2}, \tag{2.13}$$

where $\phi^0 = \phi_0 \in \mathscr{S}$. By Lemma 1 and Remark 1 $\widetilde{\phi}^{n+1}$, as the unique weak solution to (2.12), is well defined for any $\tau > 0$ when $\beta \ge 0$ and for $0 < \tau \le a_{\phi^n}$ when $\beta < 0$. In fact, when $\beta < 0$, our main result below indicates that as long as $0 < \tau \le \tau_0$ for some $\tau_0 > 0$, depending on ϕ_0, it is guaranteed that $\tau \le a_{\phi^n}$ for all $n \ge 0$ with $\{\phi^n\}_{n\in\mathbb{N}}$ generated by the GFSI (2.12)–(2.13) starting from ϕ_0. Thus, the GFSI is well defined for all $n \geq 0$. Moreover, although the operator H_{ϕ^n} is defined from $H_0^1(\Omega)$ to $H^{-1}(\Omega)$, the regularity $\phi^n \in H_0^1(\Omega)$ guarantees that $H_{\phi^n}\widetilde{\phi}^{n+1} \in H_0^1(\Omega)$. Indeed, we have

$$H_{\phi^n}\widetilde{\phi}^{n+1} = (\phi^n - \widetilde{\phi}^{n+1})/\tau \in H_0^1(\Omega).$$

As previously mentioned, the GFSI (2.12)–(2.13) consistently exhibits energy dissipation and global convergence in various numerical experiments, which has contributed to its popularity. The following theorem provides, for the first time, a rigorous proof of these properties for both focusing and defocusing cases with and without rotation.

THEOREM 1. Under the general assumptions for the existence of the ground states, given any initial function $\phi_0 \in \mathscr{S}$, there exists $\tau_0 > 0$, depending on the initial energy $E(\phi_0)$ and β such that for any $0 < \tau \leq \tau_0$ the iteration sequence $\{\phi^n\}_{n\in\mathbb{N}}$ generated by the GFSI (2.12)–(2.13) is well defined and has the following properties:

(i) Energy dissipation, i.e., $E(\phi^{n+1}) - E(\phi^n) \leq -\frac{C_0}{2}\|\phi^{n+1} - \phi^n\|_{H_0^1}^2, \quad n \geq 0.$

(ii) Global convergence, i.e., there exists a subsequence $\{\phi^{n_j}\}_{j\in\mathbb{N}} \subset \{\phi^n\}_{n\in\mathbb{N}}$ such that ϕ^{n_j} converges to some $\phi_s \in \mathscr{S}$ strongly in $H_0^1(\Omega)$, where ϕ_s is a stationary state of the GPE satisfying the following equation in the weak sense:

$$H_{\phi_s}\phi_s = \lambda_s\phi_s, \qquad \lambda_s = \langle H_{\phi_s}\phi_s, \phi_s\rangle.$$

In Theorem 1 (ii) the global convergence holds only for a subsequence. However, if there is certain uniqueness of the stationary state with energy $E_s = \lim_{n\to\infty} E(\phi^n)$ then we can establish the convergence of the entire sequence. In particular, this is true for the nonrotating and defocusing case, i.e., $\omega = 0$ and $\beta \geq 0$.

COROLLARY 1. When $\omega = 0$ and $\beta \geq 0$, under the same assumptions of Theorem 1, if $\phi_0(\mathbf{x}) \geq 0$ for $\mathbf{x} \in \Omega$ then the entire sequence $\{\phi^n\}_{n\in\mathbb{N}}$ generated by the GFSI converges to the unique positive ground state ϕ_g strongly in $H_0^1(\Omega)$.

Corollary 1 is an immediate consequence by combining Theorem 1 (ii), the uniqueness of the positive stationary state, which is also the ground state (see (Henning & Peterseim, 2020, lemma 5.4)), and the positivity-preservation of the GFSI (2.12)–(2.13) in Lemma 7.

REMARK 2. The results in Theorem 1 and Corollary 1 are stated for a constant step size τ, but they can be directly generalized to variable time steps. In this case the time step size τ at the nth iteration need to satisfy $0 < \tau \leq \tau_n$, where τ_n depends on $E(\phi^n)$ and increases as $E(\phi^n)$ decreases. Due to the energy dissipation property, larger time steps can be progressively adopted as the iteration proceeds.

3. Proof of energy dissipation and global convergence

In this section we present the proof of the main result Theorem 1 to show the energy dissipation and global convergence of the GFSI. We first introduce an equivalent formulation of the GFSI.

3.1 *An equivalent formulation of the GFSI*

By (2.13) we have $\widetilde{\phi}^{n+1} = \|\widetilde{\phi}^{n+1}\|_{L^2}\phi^{n+1}$. Plugging it into (2.12) we obtain

$$\frac{\phi^{n+1}-\phi^n}{\tau} = -H_{\phi^n}\phi^{n+1} + \left(\frac{1-\|\widetilde{\phi}^{n+1}\|_{L^2}}{\tau\|\widetilde{\phi}^{n+1}\|_{L^2}}\right)\phi^n. \tag{3.1}$$

Note that (2.12) implies $\widetilde{\phi}^{n+1} = (I+\tau H_{\phi^n})^{-1}\phi^n$ in the sense of Remark 1. Thus, we get an equivalent form of the GFSI (2.12)–(2.13) as follows:

$$\frac{\phi^{n+1}-\phi^n}{\tau} = -H_{\phi^n}\phi^{n+1} + \lambda^{n+1}\phi^n, \quad \phi^n = 0 \text{ on } \partial\Omega, \qquad n \geq 0 \tag{3.2}$$

$$\lambda^{n+1} = \Big(1 - \big\|(I+\tau H_{\phi^n})^{-1}\phi^n\big\|_{L^2}\Big)\Big/\Big(\tau\big\|(I+\tau H_{\phi^n})^{-1}\phi^n\big\|_{L^2}\Big), \tag{3.3}$$

where $\phi^0 = \phi_0 \in \mathscr{S}$. The equivalent form (3.2)–(3.3) is reminiscent of the CNGF (2.11). In fact, directly demonstrating energy dissipation from the GFSI formulation (2.12)–(2.13) can be challenging. The following lemma and discussion explains this.

LEMMA 2. Let $\beta \geq 0$ and $\tau > 0$. For $\widetilde{\phi}^{n+1}$ obtained from (2.12) with $\phi^n \in \mathscr{S}$ we have

$$\|\widetilde{\phi}^{n+1}\|_{L^2} \leq 1 \quad \text{and} \quad E(\widetilde{\phi}^{n+1}) \leq E(\phi^{n+1}). \tag{3.4}$$

Proof. Testing (2.12) with $\tau\widetilde{\phi}^{n+1}$, using (2.8) and Cauchy's inequality and noting $\beta \geq 0$ we obtain

$$\|\widetilde{\phi}^{n+1}\|_{L^2}^2 = (\phi^n, \widetilde{\phi}^{n+1})_{L^2} - \tau\langle H_{\phi^n}\widetilde{\phi}^{n+1}, \widetilde{\phi}^{n+1}\rangle \leq \|\phi^n\|_{L^2}\|\widetilde{\phi}^{n+1}\|_{L^2},$$

which implies $\|\widetilde{\phi}^{n+1}\|_{L^2} \leq \|\phi^n\|_{L^2} = 1$. For the energy we have

$$\begin{aligned} E(\widetilde{\phi}^{n+1}) = E(\phi^{n+1}\|\widetilde{\phi}^{n+1}\|_{L^2}) &= \|\widetilde{\phi}^{n+1}\|_{L^2}^2\langle H_0\phi^{n+1}, \phi^{n+1}\rangle + \frac{\beta}{2}\|\widetilde{\phi}^{n+1}\|_{L^2}^4\|\phi^{n+1}\|_{L^4}^4 \\ &\leq \langle H_0\phi^{n+1}, \phi^{n+1}\rangle + \frac{\beta}{2}\|\phi^{n+1}\|_{L^4}^4 = E(\phi^{n+1}), \end{aligned}$$

which completes the proof. □

A natural approach to establish the energy dissipation of the GFSI, which has already been attempted in the literature Bao & Du (2004), is to first establish that $E(\widetilde{\phi}^{n+1}) \leq E(\phi^n)$ (or for some modified energy) and subsequently that $E(\phi^{n+1}) \leq E(\phi^n)$. However, Lemma 2 indicates that this approach may be more challenging due to the fact $E(\widetilde{\phi}^{n+1}) \leq E(\phi^{n+1})$ for $\beta > 0$. Hence, to obtain the energy dissipation of the GFSI (2.12)–(2.13), we reformulate the GFSI into (3.2)–(3.3) in terms of ϕ^{n+1} and ϕ^n without using the intermediary function $\widetilde{\phi}^{n+1}$. As a result, we are able to establish the proof of Theorem 1 as to be presented in the following.

3.2 *Proof of Theorem* 1 (i): *energy dissipation*

In this subsection we prove Theorem 1 (*i*) for the energy dissipation. Before presenting the proof we introduce some auxiliary estimates as follows.

First, for λ^{n+1} in (3.3), we have the following estimate.

LEMMA 3. Let λ^{n+1} be given by (3.3). For any $\phi^n \in \mathscr{S}$ and $0 < \tau < a_{\phi^n}$ with a_{ϕ^n} given in Lemma 1 (iii) we have $\lambda^{n+1} \geq 0$ if $\beta \geq 0$ and $\lambda^{n+1} \geq -1/\tau$ if $\beta < 0$.

Proof. By Lemma 1 (*ii*) and (*iii*) $(I + \tau H_{\phi^n})^{-1}\phi^n \in H_0^1(\Omega)$ is well defined under given assumptions. Then, the estimate for $\beta \geq 0$ follows immediately from Lemma 2 and (3.3). The estimate for $\beta < 0$ is also an immediate result of (3.3). □

In addition the H_0^1-norm of $\phi \in \mathscr{S}$ can be bounded by the energy as follows.

LEMMA 4. For any $\phi \in \mathscr{S}$ we have

(i) when $\beta \geq 0$, $E(\phi) \geq \gamma \|\phi\|^2_{H_0^1}$ with $\gamma = C_0$;

(ii) when $d = 2$ and $-C_b < \beta < 0$, $E(\phi) \geq \gamma \|\phi\|^2_{H_0^1}$ with $\gamma = \frac{1}{2}\min\left\{\frac{K}{(1+K)}, 1 + \frac{\beta}{C_b}\right\}$;

(iii) when $d = 1$ and $\beta < 0$, $E(\phi) + C_\beta \geq \frac{1}{4}\|\phi\|^2_{H_0^1}$ with $C_\beta := C\beta^2$.

Proof. For (*i*), by $\beta \geq 0$ and (2.8) we have

$$\begin{aligned} E(\phi) &= \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + V(\mathbf{x})|\phi|^2 + \frac{\beta}{2}|\phi|^4 - \omega\overline{\phi}L_z\phi\right) \mathrm{d}\mathbf{x} \\ &\geq \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + V(\mathbf{x})|\phi|^2 - \omega\overline{\phi}L_z\phi\right) \mathrm{d}\mathbf{x} \\ &= \langle H_0\phi, \phi\rangle \geq C_0\|\phi\|^2_{H_0^1}. \end{aligned}$$

For (ii), noting that, for any $\widetilde{\omega} \in \mathbb{R}$,

$$|\nabla\phi|^2 - 2\mathrm{Re}\left(\widetilde{\omega}\overline{\phi}L_z\phi\right) = |(\nabla - i\widetilde{\omega}A(\mathbf{x}))\,\phi|^2 - \widetilde{\omega}^2|\mathbf{x}|^2|\phi|^2, \tag{3.5}$$

where $A(\mathbf{x}) = (-y, x)^{\mathrm{T}}$. By the pointwise diamagnetic inequality (Lieb & Loss, 2001) we have

$$|\nabla|\phi|(\mathbf{x})| \leq |(\nabla - i\,\widetilde{\omega}A)\,\phi(\mathbf{x})|, \quad \text{for a.e. } \mathbf{x} \in \Omega. \tag{3.6}$$

From (2.1), by (3.5) with $\widetilde{\omega} = \omega/\varepsilon$ we have

$$\begin{aligned} E(\phi) &= \int_\Omega \left[\frac{1-\varepsilon}{2}|\nabla\phi|^2 + \frac{\varepsilon}{2}|\nabla\phi|^2 - \frac{\varepsilon}{2}2\mathrm{Re}\left(\frac{\omega}{\varepsilon}\overline{\phi}L_z\phi\right) + V|\phi|^2 + \frac{\beta}{2}|\phi|^4\right] \mathrm{d}\mathbf{x} \\ &= \int_\Omega \left[\frac{\varepsilon}{2}\left|\left(\nabla - i\frac{\omega}{\varepsilon}A\right)\phi\right|^2 - \frac{\omega^2}{2\varepsilon}|\mathbf{x}|^2|\phi|^2 + V|\phi|^2\right] \mathrm{d}\mathbf{x} + \frac{1-\varepsilon}{2}\|\phi\|^2_{H_0^1} + \frac{\beta}{2}\|\phi\|^4_{L^4}. \end{aligned}$$

Moreover, by (3.6) with $\widetilde{\omega} = \omega/\varepsilon$, recalling (2.5) and $\|\phi\|_{L^2} = 1$ we get

$$E(\phi) \geq \int_\Omega \left[\frac{\varepsilon}{2}\,|\nabla|\phi||^{\,2} + \left(V(\mathbf{x}) - \frac{\omega^2}{2\varepsilon}|\mathbf{x}|^2\right)|\phi|^2\right] \mathrm{d}\mathbf{x} + \frac{1-\varepsilon}{2}\|\phi\|^2_{H_0^1} + \frac{\beta}{2C_b}\|\nabla|\phi|\|^2_{L^2}$$
$$= \int_\Omega \left(V(\mathbf{x}) - \frac{\omega^2}{2\varepsilon}|\mathbf{x}|^2\right)|\phi|^2\,\mathrm{d}\mathbf{x} + \frac{1-\varepsilon}{2}\|\phi\|^2_{H_0^1} + \left(\frac{\varepsilon}{2} + \frac{\beta}{2C_b}\right)\|\nabla|\phi|\|^2_{L^2}.$$

Taking $\varepsilon = \max\left\{1/(1+K), -\beta/C_b\right\} < 1$ in the above inequality, noting that

$$V(\mathbf{x}) \geq \frac{(1+K)\omega^2}{2}|\mathbf{x}|^2 \geq \frac{\omega^2}{2\varepsilon}|\mathbf{x}|^2 \qquad \text{and} \qquad \frac{\varepsilon}{2} \geq -\frac{\beta}{2C_b},$$

we get $E(\phi) \geq \frac{1-\varepsilon}{2}\|\phi\|^2_{H_0^1} =: \gamma\|\phi\|^2_{H_0^1}$, with $\gamma = (1-\varepsilon)/2 = \min\left\{K/(1+K), 1+\beta/C_b\right\}/2$.

For (iii), by the Gagliardo–Nirenberg inequality, noting $\|\phi\|_{L^2} = 1$, we have

$$E(\phi) = \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + V|\phi|^2 + \frac{\beta}{2}|\phi|^4\right) \mathrm{d}\mathbf{x}$$
$$\geq \frac{1}{2}\|\phi\|^2_{H_0^1} + C\beta\|\phi\|^3_{L^2}\|\phi\|_{H_0^1}$$
$$\geq \frac{1}{2}\|\phi\|^2_{H_0^1} - \frac{1}{4}\|\phi\|^2_{H_0^1} - C^2\beta^2,$$

which implies

$$E(\phi) + C^2\beta^2 \geq \frac{1}{4}\|\phi\|^2_{H_0^1}.$$

The proof is completed. □

Finally, we can bound the H_0^1-norm of ϕ^{n+1} in (3.2)–(3.3) by the energy at the previous step.

LEMMA 5. Let $\beta \geq 0$ and $\phi^n \in \mathscr{S}$. For ϕ^{n+1} obtained from (3.2)–(3.3) we have

$$\|\phi^{n+1}\|_{H_0^1} \leq C_{E^n}, \qquad 0 < \tau \leq 1/(4E(\phi^n)),$$

where $C_{E^n} := C\sqrt{E(\phi^n)}$.

Proof. Since $\phi^n, \widetilde{\phi}^{n+1} \in H_0^1(\Omega)$ we have $H_{\phi^n}\widetilde{\phi}^{n+1} \in H_0^1(\Omega)$ by (2.12). Taking the L^2-inner product with $\tau H_{\phi^n}\widetilde{\phi}^{n+1}$ on both sides of (2.12), using Lemma 1 (*i*), we get

$$\begin{aligned} C_0\|\widetilde{\phi}^{n+1}\|_{H_0^1}^2 &\le \langle H_{\phi^n}\widetilde{\phi}^{n+1}, \widetilde{\phi}^{n+1}\rangle = (\widetilde{\phi}^{n+1}, H_{\phi^n}\widetilde{\phi}^{n+1})_{L^2} \\ &= (\phi^n, H_{\phi^n}\widetilde{\phi}^{n+1})_{L^2} - \tau(H_{\phi^n}\widetilde{\phi}^{n+1}, H_{\phi^n}\widetilde{\phi}^{n+1})_{L^2} \\ &\le (\phi^n, H_{\phi^n}\widetilde{\phi}^{n+1})_{L^2} = \langle H_{\phi^n}\phi^n, \widetilde{\phi}^{n+1}\rangle. \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \widetilde{\phi}^{n+1} &= (I+\tau H_{\phi^n})^{-1}\phi^n = (I+\tau H_{\phi^n})^{-1}(I+\tau H_{\phi^n} - \tau H_{\phi^n})\phi^n \\ &= \phi^n - \tau(I+\tau H_{\phi^n})^{-1}H_{\phi^n}\phi^n =: \phi^n - \tau w^n, \end{aligned}$$

which plugged into (3.7) yields

$$C_0\|\widetilde{\phi}^{n+1}\|_{H_0^1}^2 \le \langle H_{\phi^n}\phi^n, \phi^n\rangle - \tau\langle H_{\phi^n}\phi^n, w^n\rangle \le \langle H_{\phi^n}\phi^n, \phi^n\rangle \le 2E(\phi^n), \tag{3.8}$$

where we use $\langle H_{\phi^n}\phi^n, \phi^n\rangle \le 2E(\phi^n)$ and $\langle H_{\phi^n}\phi^n, w^n\rangle = \mathscr{A}_{\phi^n}^{\tau}(w^n, w^n) \ge 0$. This indicates that

$$\|\widetilde{\phi}^{n+1}\|_{H_0^1} \le \sqrt{\frac{2}{C_0}E(\phi^n)}. \tag{3.9}$$

Recalling (2.13) it remains to obtain a lower bound of $\|\widetilde{\phi}^{n+1}\|_{L^2}$. Taking the L^2-inner product with $\tau\phi^n$ on both sides of (2.12), recalling again $\|\phi^n\|_{L^2} = 1$, we obtain

$$\begin{aligned} \|\widetilde{\phi}^{n+1}\|_{L^2} \ge (\widetilde{\phi}^{n+1}, \phi^n)_{L^2} &= 1 - \tau(H_{\phi^n}\widetilde{\phi}^{n+1}, \phi^n)_{L^2} = 1 - \tau\langle H_{\phi^n}\widetilde{\phi}^{n+1}, \phi^n\rangle \\ &= 1 - \tau\langle H_{\phi^n}\phi^n, \phi^n\rangle + \tau^2\langle H_{\phi^n}w^n, \phi^n\rangle \ge 1 - 2\tau E(\phi^n), \end{aligned} \tag{3.10}$$

where we applied $\langle H_{\phi^n}\phi^n, \phi^n\rangle \le 2E(\phi^n)$ and $\langle H_{\phi^n}w^n, \phi^n\rangle = \langle H_{\phi^n}\phi^n, w^n\rangle = \mathscr{A}_{\phi^n}^{\tau}(w^n, w^n) \ge 0$. Hence, when $\tau \le 1/(4E(\phi^n))$, we have $\|\widetilde{\phi}^{n+1}\|_{L^2} \ge 1/2$. Then, by (3.9),

$$\|\phi^{n+1}\|_{H_0^1} = \frac{\|\widetilde{\phi}^{n+1}\|_{H_0^1}}{\|\widetilde{\phi}^{n+1}\|_{L^2}} \le 2\sqrt{\frac{2}{C_0}E(\phi^n)} = C\sqrt{E(\phi^n)} = C_{E^n}.$$

The proof is thus completed. □

With the above lemmas we are now ready to prove Theorem 1 (i).

Proof of Theorem 1 (i): We only present the proof for the nonlinear case, i.e., $\beta \ne 0$. The linear case, i.e., $\beta = 0$, is an immediate consequence and we can set $\tau_0 = +\infty$ in this case. In the following equation we first consider the defocusing case $\beta > 0$. Testing (3.2) with $2(\phi^{n+1} - \phi^n)$ we get

$$\frac{2}{\tau}\|\phi^{n+1} - \phi^n\|^2_{L^2(\Omega)} = -2\,\langle H_{\phi^n}\phi^{n+1}, \phi^{n+1} - \phi^n\rangle + 2\lambda^{n+1}(\phi^n, \phi^{n+1} - \phi^n)_{L^2}$$
$$=: I_1 + I_2. \tag{3.11}$$

For I_1, by the symmetry of $\langle H_{\phi^n}\cdot, \cdot\rangle$, we have

$$I_1 = \langle H_{\phi^n}\phi^n, \phi^n\rangle - \langle H_{\phi^n}\phi^{n+1}, \phi^{n+1}\rangle - \langle H_{\phi^n}(\phi^{n+1} - \phi^n), \phi^{n+1} - \phi^n\rangle. \tag{3.12}$$

For I_2, noticing that $\|\phi^n\|_{L^2} = \|\phi^{n+1}\|_{L^2} = 1$, we get

$$I_2 = \lambda^{n+1}\left[\|\phi^{n+1}\|^2_{L^2} - \|\phi^n\|^2_{L^2} - \|\phi^{n+1} - \phi^n\|^2_{L^2}\right] = -\lambda^{n+1}\|\phi^{n+1} - \phi^n\|^2_{L^2}. \tag{3.13}$$

Plugging (3.12) and (3.13) into (3.11), noticing (2.8), we derive

$$\begin{aligned}
&\langle H_{\phi^n}\phi^{n+1}, \phi^{n+1}\rangle - \langle H_{\phi^n}\phi^n, \phi^n\rangle \\
&\quad = -\frac{2}{\tau}\|\phi^{n+1} - \phi^n\|^2_{L^2} - \lambda^{n+1}\|\phi^{n+1} - \phi^n\|^2_{L^2} - \langle H_{\phi^n}(\phi^{n+1} - \phi^n), \phi^{n+1} - \phi^n\rangle \\
&\quad \le -\left(\frac{2}{\tau} + \lambda^{n+1}\right)\|\phi^{n+1} - \phi^n\|^2_{L^2} - C_0\|\phi^{n+1} - \phi^n\|^2_{H^1_0} - \beta\int_\Omega |\phi^n|^2|\phi^{n+1} - \phi^n|^2\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{3.14}$$

Recalling the definition of E (2.1) we have

$$E(\phi^{n+1}) = \langle H_{\phi^n}\phi^{n+1}, \phi^{n+1}\rangle - \beta\int_\Omega |\phi^n|^2|\phi^{n+1}|^2\,\mathrm{d}\mathbf{x} + \frac{\beta}{2}\int_\Omega |\phi^{n+1}|^4\,\mathrm{d}\mathbf{x}, \tag{3.15}$$

$$E(\phi^n) = \langle H_{\phi^n}\phi^n, \phi^n\rangle - \frac{\beta}{2}\int_\Omega |\phi^n|^4\,\mathrm{d}\mathbf{x}. \tag{3.16}$$

Subtracting (3.16) from (3.15), by (3.14) we obtain

$$\begin{aligned}
E(\phi^{n+1}) - E(\phi^n) &= \langle H_{\phi^n}\phi^{n+1}, \phi^{n+1}\rangle - \langle H_{\phi^n}\phi^n, \phi^n\rangle + \frac{\beta}{2}\int_\Omega \left||\phi^{n+1}|^2 - |\phi^n|^2\right|^2\mathrm{d}\mathbf{x} \\
&\le -\left(\frac{2}{\tau} + \lambda^{n+1}\right)\|\phi^{n+1} - \phi^n\|^2_{L^2} - C_0\|\phi^{n+1} - \phi^n\|^2_{H^1_0} \\
&\quad + \underbrace{\frac{\beta}{2}\int_\Omega \left||\phi^{n+1}|^2 - |\phi^n|^2\right|^2\mathrm{d}\mathbf{x} - \beta\int_\Omega |\phi^n|^2|\phi^{n+1} - \phi^n|^2\,\mathrm{d}\mathbf{x}}_{=:I_3}.
\end{aligned} \tag{3.17}$$

By the Hölder's inequality, Gargliardo–Nirenberg inequality, Sobolev embedding $H^1 \hookrightarrow L^4$ and Lemma 5, recalling $\|\phi^{n+1}\|_{L^2} = 1$, we have

$$
\begin{aligned}
I_3 &\le \frac{\beta}{2}\int_\Omega \left(|\phi^{n+1}| + |\phi^n|\right)^2 |\phi^{n+1} - \phi^n|^2 \,\mathrm{d}\mathbf{x} - \beta\int_\Omega |\phi^n|^2|\phi^{n+1} - \phi^n|^2 \,\mathrm{d}\mathbf{x} \\
&\le \beta\int_\Omega \left(|\phi^{n+1}|^2 + |\phi^n|^2\right) |\phi^{n+1} - \phi^n|^2 \,\mathrm{d}\mathbf{x} - \beta\int_\Omega |\phi^n|^2|\phi^{n+1} - \phi^n|^2 \,\mathrm{d}\mathbf{x} \\
&= \beta\int_\Omega |\phi^{n+1}|^2|\phi^{n+1} - \phi^n|^2 \,\mathrm{d}\mathbf{x} \\
&\le \beta\|\phi^{n+1}\|_{L^4}^2\|\phi^{n+1} - \phi^n\|_{L^4}^2 \\
&\le C\beta\|\phi^{n+1}\|_{L^2}^{2-d/2}\|\phi^{n+1}\|_{H_0^1}^{d/2}\|\phi^{n+1} - \phi^n\|_{L^2}^{2-d/2}\|\phi^{n+1} - \phi^n\|_{H_0^1}^{d/2} \\
&\le C\beta C_{E^n}^{d/2}\left(\varepsilon^{-\frac{d}{4-d}}\|\phi^{n+1} - \phi^n\|_{L^2}^2 + \varepsilon\|\phi^{n+1} - \phi^n\|_{H_0^1}^2\right),
\end{aligned}
\tag{3.18}
$$

where we use the Young's inequality $ab \le \varepsilon^{-p/q}a^p + \varepsilon b^q$ with $p = 4/(4-d)$ and $q = 4/d$ in the last line. From (3.18), choosing $\varepsilon = C_0/(2C\beta C_{E^n}^{d/2})$, we get

$$
I_3 \le \widetilde{C}_{E^n}\|\phi^{n+1} - \phi^n\|_{L^2}^2 + \frac{C_0}{2}\|\phi^{n+1} - \phi^n\|_{H_0^1}^2, \tag{3.19}
$$

where $\widetilde{C}_{E^n} := C\beta C_{E^n}^{d/2}\left(2C\beta C_{E^n}^{d/2}/C_0\right)^{\frac{d}{4-d}} = C(\beta^2 C_{E^n}^d)^{\frac{2}{4-d}}$ increases as $E(\phi^n)$ increases. Inserting (3.19) into (3.17), recalling $\lambda^{n+1} \ge 0$ when $\beta > 0$ by Lemma 3, we obtain

$$
E(\phi^{n+1}) - E(\phi^n) \le \left(-\frac{2}{\tau} + \widetilde{C}_{E^n}\right)\|\phi^{n+1} - \phi^n\|_{L^2}^2 - \frac{C_0}{2}\|\phi^{n+1} - \phi^n\|_{H_0^1}^2.
$$

Hence, choosing τ such that $-2/\tau + \widetilde{C}_{E^n} \le 0$, i.e., $\tau \le 2/\widetilde{C}_{E^n}$, leads to the energy dissipation law at the nth step: let $\tau_n := \min\{1/(4E(\phi^n)), 2/\widetilde{C}_{E^n}\}$,

$$
E(\phi^{n+1}) - E(\phi^n) \le -\frac{C_0}{2}\|\phi^{n+1} - \phi^n\|_{H_0^1}^2, \qquad 0 < \tau \le \tau_n. \tag{3.20}
$$

Finally, we use an induction argument to show that the conclusion holds when

$$
0 < \tau \le \tau_0 := \min\left\{\frac{1}{4E(\phi^0)}, \frac{2}{\widetilde{C}_{E^0}}\right\}.
$$

By (3.20), it suffices to show that $\tau_0 \le \tau_n$ for all $n \ge 0$, which reduces to showing that the energy is decaying, i.e., $E(\phi^{n+1}) \le E(\phi^n)$ for all $n \ge 0$. When $n = 0$, by (3.20) and $0 < \tau \le \tau_0$, we have $E(\phi^1) \le E(\phi^0)$. We now assume that $E(\phi^{n+1}) \le E(\phi^n)$ for all $0 \le n \le m-1$. Consequently, we have $0 \le E(\phi^m) \le E(\phi^{m-1}) \le \cdots \le E(\phi^0)$, which implies $\tau_0 \le \tau_1 \le \cdots \le \tau_m$. In particular, we have $0 < \tau \le \tau_m$. Then, by (3.20), the assumption holds for $n = m$, and thus for all $n \ge 0$ by the mathematical induction. This concludes the proof of the energy dissipation for $\beta > 0$.

Then, we consider the focusing case $\beta < 0$. For the GFSI (2.12)–(2.13) to be well defined at the nth step, according to Lemma 1 (iii) and Lemma 4 (ii), the following step size restriction need to be satisfied:

$$\begin{cases} 0 < \tau \le \frac{\gamma C}{\beta^2 E(\phi^n)} \le a_{\phi^n}, & d = 2, \\ 0 < \tau \le \frac{C}{4^{\frac{1}{3}} |\beta|^{\frac{4}{3}} (E(\phi^n) + C_\beta)^{\frac{1}{3}}} \le a_{\phi^n}, & d = 1. \end{cases} \tag{3.21}$$

Note that (3.17) remains valid for $\beta < 0$, and the estimate of I_3 for $\beta < 0$ follows exactly the same procedure outlined in (A.7):

$$\begin{aligned} I_3 &\le |\beta| \int_\Omega |\phi^n|^2 |\phi^{n+1} - \phi^n|^2 \, \mathrm{d}\mathbf{x} \\ &\le |\beta| \|\phi^n\|_{L^4}^2 \|\phi^{n+1} - \phi^n\|_{L^4}^2 \\ &\le \frac{1}{a_{\phi^n}} \|\phi^{n+1} - \phi^n\|_{L^2}^2 + \frac{C_0}{2} \|\phi^{n+1} - \phi^n\|_{H_0^1}^2, \end{aligned} \tag{3.22}$$

which plugged into (3.17), by recalling $\lambda^{n+1} \ge -1/\tau$ when $\beta < 0$ in Lemma 3, yields

$$E(\phi^{n+1}) - E(\phi^n) \le \left(-\frac{1}{\tau} + \frac{1}{a_{\phi^n}} \right) \|\phi^{n+1} - \phi^n\|_{L^2}^2 - \frac{C_0}{2} \|\phi^{n+1} - \phi^n\|_{H_0^1}^2. \tag{3.23}$$

By the step size restriction $\tau \le a_{\phi^n}$ in (3.21) we have

$$E(\phi^{n+1}) - E(\phi^n) \le -\frac{C_0}{2} \|\phi^{n+1} - \phi^n\|_{H_0^1}^2. \tag{3.24}$$

Hence, by a similar induction argument as in the defocusing case, we prove the energy dissipation for the focusing case $\beta < 0$ under the step size restriction

$$0 < \tau \le \tau_0 := \begin{cases} \frac{\gamma C}{\beta^2 E(\phi^0)}, & d = 2, \\ \frac{C}{4^{\frac{1}{3}} |\beta|^{\frac{4}{3}} (E(\phi^0) + C_\beta)^{\frac{1}{3}}}, & d = 1, \end{cases} \tag{3.25}$$

which completes the proof. □

3.3 *Proof of Theorem* 1 (ii): *global convergence*

In this section we prove the global convergence of the GFSI. We first introduce some additional lemmas.

Lemma 6. Let $\phi \in \mathscr{S}$ and $0 < a \le a_\phi$. For any $v \in L^q(\Omega)$ with $6/5 \le q \le \infty$ we have

$$\|(I + aH_\phi)^{-1} v\|_{H_0^1} \le (C/a) \|v\|_{L^q},$$

where C is a constant that depends on Ω, d and K, but is independent of ϕ and v.

Proof. First, by Remark 1, $(1 + aH_\phi)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is well defined when $0 < a \le a_\phi$. By Lemma 1 (iii), Hölder's inequality and Sobolev embedding we have

$$\begin{aligned} a\|(I + aH_\phi)^{-1}v\|^2_{H_0^1} &\le C\mathscr{A}^a_\phi\big((I + aH_\phi)^{-1}v, (I + aH_\phi)^{-1}v\big) \\ &= C\left(v, (I + aH_\phi)^{-1}v\right)_{L^2} \\ &\le C\big\|(I + aH_\phi)^{-1}v\big\|_{L^p}\|v\|_{L^q} \\ &\le C\big\|(I + aH_\phi)^{-1}v\big\|_{H_0^1}\|v\|_{L^q}, \end{aligned}$$

where $q = p/(p-1)$ and $1 \le p \le 6$, and C depends on Ω, d and K. □

To prove Corollary 1 we need the following result, which implies the positivity-preserving property of the GFSI (2.12)–(2.13).

LEMMA 7. Let $\omega = 0$, $\beta \ge 0$, $\phi \in \mathscr{S}$ and $a > 0$. For any $v \in H_0^1(\Omega)$ satisfying $v(\mathbf{x}) \ge 0$ for a.e. $\mathbf{x} \in \Omega$ we have $(I + aH_\phi)^{-1}v(\mathbf{x}) \ge 0$ for a.e. $\mathbf{x} \in \Omega$.

Proof. Again, $(I+aH_\phi)^{-1}v$ is well defined under the given assumptions. We define a quadratic functional $J : H_0^1(\Omega) \to \mathbb{R}$ as

$$\begin{aligned} J(u) &:= \frac{1}{2}\mathscr{A}^a_\phi(u, u) - (v, u)_{L^2} \\ &= \frac{1}{2}\|u\|^2_{L^2} + \frac{a}{2}\int_\Omega \left(\frac{1}{2}|\nabla u| + V|u|^2 + \beta|\phi|^2|u|^2\right) \mathrm{d}\mathbf{x} - \mathrm{Re}\int_\Omega v\bar{u}\,\mathrm{d}x. \end{aligned}$$

By Lemma 1 (iii), $J''(u)[\cdot,\cdot] = \mathscr{A}^a_\phi(\cdot,\cdot)$ is positive definite, and thus J is strongly convex on $H_0^1(\Omega)$. Hence, J has a unique minimizer $\widetilde{u}$ on $H_0^1(\Omega)$, which satisfies

$$\mathscr{A}^a_\phi(\widetilde{u}, w) = (v, w), \qquad \text{for all } w \in H_0^1(\Omega).$$

Recalling Remark 1, we have $(I + \tau H_\phi)^{-1}v = \widetilde{u}$. On the other hand, since

$$|\nabla|\widetilde{u}|(\mathbf{x})| \le |\nabla\widetilde{u}(\mathbf{x})|, \quad v(\mathbf{x})|\widetilde{u}(\mathbf{x})| \ge \mathrm{Re}(v(\mathbf{x})\overline{\widetilde{u}(\mathbf{x})}), \qquad \text{for a.e. } \mathbf{x} \in \Omega,$$

we have $J(\widetilde{u}) \ge J(|\widetilde{u}|)$, and thus $|\widetilde{u}|$ is also a minimizer of J on $H_0^1(\Omega)$. The uniqueness of the minimizer then implies that $\widetilde{u} = |\widetilde{u}|$ a.e. in Ω, which completes the proof. □

Proof of Theorem 1 (ii): By the monotonicity of $E(\phi^n)$ obtained in Theorem 1 (i) and that E is bounded below on $\mathscr{S}$ there exists $E_\mathrm{s} \in \mathbb{R}$ such that

$$\lim_{n\to\infty} E(\phi^n) = E_\mathrm{s}.$$

By Theorem 1 (i) we also have

$$\lim_{n\to\infty} \|\phi^{n+1} - \phi^n\|_{H_0^1} \le \lim_{n\to\infty} \sqrt{\frac{2}{C_0}(E(\phi^n) - E(\phi^{n+1}))} = 0. \tag{3.26}$$

In addition, by Lemma 4 and the boundedness of the energy sequence $E(\phi^n)_{n\in\mathbb{N}}$, the iteration sequence $\{\phi^n\}_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Hence, there exists $\phi_\mathrm{s} \in H_0^1(\Omega)$ and a subsequence $\{\phi^{n_j}\}_{j\in\mathbb{N}}$ such that

$$\phi^{n_j} \rightharpoonup \phi_\mathrm{s}, \qquad \text{weakly in } H_0^1(\Omega). \tag{3.27}$$

By the compact embedding $H_0^1(\Omega) \subset\subset L^p(\Omega)$ $(1 \le p < 6)$ when $d \le 3$ we have

$$\phi^{n_j} \to \phi_\mathrm{s}, \qquad \text{strongly in } L^p(\Omega)\ (1 \le p < 6). \tag{3.28}$$

In particular, we have $\|\phi_s\|_{L^2} = \lim_{j\to\infty} \|\phi^{n_j}\|_{L^2} = 1$, and thus $\phi_\mathrm{s} \in \mathscr{S}$.

Define $\widetilde{\phi}_\mathrm{s} := (1 + \tau H_{\phi_s})^{-1}\phi_s \in H_0^1(\Omega)$. We then show that

$$\widetilde{\phi}^{n_j+1} = (1 + \tau H_{\phi^{n_j}})^{-1}\phi^{n_j} \to \widetilde{\phi}_s, \qquad \text{strongly in } H_0^1(\Omega) \text{ as } j \to \infty. \tag{3.29}$$

Note that

$$\widetilde{\phi}^{n_j+1} - \widetilde{\phi}_s = (I + \tau H_{\phi^{n_j}})^{-1}(\phi^{n_j} - \phi_s) + \Big((I + \tau H_{\phi^{n_j}})^{-1} - (I + \tau H_{\phi_s})^{-1}\Big)\phi_s =: W_1 + W_2. \tag{3.30}$$

For W_1, by Lemma 6 with $q = 2$, we have

$$\|W_1\|_{H_0^1} \le (C/\tau)\|\phi^{n_j} - \phi_s\|_{L^2}. \tag{3.31}$$

For W_2 direct calculation yields that

$$\begin{aligned} W_2 &= \Big((I + \tau H_{\phi^{n_j}})^{-1}(I + \tau H_{\phi_s} - I - \tau H_{\phi^{n_j}})(I + \tau H_{\phi_s})^{-1}\Big)\phi_s \\ &= \tau\beta(I + \tau H_{\phi^{n_j}})^{-1}\Big((|\phi_s|^2 - |\phi^{n_j}|^2)\widetilde{\phi}_s\Big). \end{aligned} \tag{3.32}$$

From (3.32), using Lemma 6, Hölder's inequality and Sobolev embedding, we have

$$\begin{aligned} \|W_2\|_{H_0^1} &\le \tau|\beta|(C/\tau)\|(|\phi_s|^2 - |\phi^{n_j}|^2)\widetilde{\phi}_s\|_{L^{4/3}} \\ &\le C|\beta|\||\phi_s|^2 - |\phi^{n_j}|^2\|_{L^2}\|\widetilde{\phi}_s\|_{L^4} \\ &\le C|\beta|\||\phi_s| + |\phi^{n_j}|\|_{L^4}\|\phi_s - \phi^{n_j}\|_{L^4}\|\widetilde{\phi}_s\|_{H_0^1} \\ &\le C|\beta|(\|\phi_s\|_{H_0^1} + \|\phi^{n_j}\|_{H_0^1})\|\widetilde{\phi}_s\|_{H_0^1}\|\phi_s - \phi^{n_j}\|_{L^4}. \end{aligned} \tag{3.33}$$

Combining (3.31) and (3.33), noting (3.30), recalling (3.28) and that ϕ^n is uniformly bounded in $H_0^1(\Omega)$ we prove (3.29). As a result of (3.29) we further have

$$\phi^{n_j+1} = \frac{\widetilde{\phi}^{n_j+1}}{\|\widetilde{\phi}^{n_j+1}\|_{L^2}} \to \frac{\widetilde{\phi}_s}{\|\widetilde{\phi}_s\|_{L^2}}, \qquad \text{strongly in } H_0^1(\Omega) \text{ as } j \to \infty.$$

By (3.26) we get ϕ^{n_j} converges strongly in $H_0^1(\Omega)$ and shares the same limit as ϕ^{n_j+1} as $j \to \infty$, i.e.,

$$\phi^{n_j} \to \frac{\widetilde{\phi}_s}{\|\widetilde{\phi}_s\|_{L^2}}, \qquad \text{strongly in } H_0^1(\Omega) \text{ as } j \to \infty,$$

which, by recalling (3.27), implies $\phi_s = \widetilde{\phi}_s / \|\widetilde{\phi}_s\|_{L^2}$. Recalling that $\widetilde{\phi}_s = (1 + \tau H_{\phi_s})^{-1}\phi_s$ we have

$$H_{\phi_s}\phi_s = \frac{1 - \|\widetilde{\phi}_s\|_{L^2}}{\tau\|\widetilde{\phi}_s\|_{L^2}}\phi_s, \qquad \text{in } H^{-1}(\Omega), \tag{3.34}$$

where we certainly have $(1 - \|\widetilde{\phi}_s\|_{L^2})/(\tau\|\widetilde{\phi}_s\|_{L^2}) = \langle H_{\phi_s}\phi_s, \phi_s\rangle = \lambda_s$ and this completes the proof. □

4. Numerical experiments

In this section we numerically justify the results in Theorem 1, i.e., the GFSI algorithm (2.12)–(2.13) has an energy-diminishing property when time step $\tau \le \tau_0$. Such τ_0 is independent of the spatial mesh size, but depends on the interaction strength β. To this end, we consider the minimization problem (1.4) on a two-dimensional domain $\Omega = [-L, L]^2$ with the harmonic trapping potential $V(\mathbf{x}) = |\mathbf{x}|^2/2$.

To numerically solve this problem we further discretize the domain Ω using a uniform grid with equal spacing in both x and y directions, i.e., $h_x = h_y = h$, and approximate the first and second derivative operators in (2.12)–(2.13) with central differences. The iteration of the resulting full-discretized GFSI algorithm is terminated whenever the following condition is fulfilled:

$$\frac{\|\phi^{n+1} - \phi^n\|_\infty}{\tau} < 10^{-7},$$

and the resulting ϕ^{n+1} is regarded as the ground state ϕ_g. Moreover, we define the related residual as

$$r_{\phi_g} := \left\|H_{\phi_g}\phi_g - \lambda_{\phi_g}\phi_g\right\|_\infty \qquad \text{with} \qquad \lambda_{\phi_g} = \left\langle H_{\phi_g}\phi_g, \phi_g\right\rangle,$$

which is used to verify the first-order necessary condition for a stationary state of the minimization problem (1.4).

Example 1. Here, we test the energy-diminishing property of the GFSI algorithm w.r.t. time step τ and mesh size h. To this end, we let $L = 8$, $\omega = 0.5$ and consider both the focusing and defocusing case:

(i) Focusing: let $\beta = -5$ and choose initial data for GFSI algorithm as $\phi_0 = \frac{e^{-|\mathbf{x}|^2/2}}{\sqrt{\pi}}$.

(ii) Defocusing: let $\beta = 100$ and choose initial data for GFSI algorithm as $\phi_0 = \frac{(x+iy)\, e^{-|\mathbf{x}|^2/2}}{\sqrt{\pi}}$.

TABLE 1 *Total energy $E(\phi_g)$, residual r_{ϕ_g} of the converged ground state ϕ_g and iteration numbers 'Iter' the GFSI method needed to converged for $\beta = -5$ (upper) and $\beta = 100$ (lower) in Example 1*

		$\tau = 0.1$		$\tau = 0.1/2$		$\tau = 0.1/4$		$\tau = 0.1/8$	
h	$E(\phi_g)$	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter
$1/2^3$	0.3933	1.8e−7	200	1.4e−7	354	1.2e−7	662	1.1e−7	1279
$1/2^4$	0.4009	1.6e−7	184	1.3e−7	326	1.2e−7	611	1.1e−7	1181
$1/2^5$	0.4027	1.7e−7	180	1.4e−7	320	1.2e−7	601	1.1e−7	1162
$1/2^6$	0.4032	1.8e−7	179	1.3e−7	319	1.2e−7	598	1.1e−7	1158
$1/2^7$	0.4033	1.7e−7	179	1.3e−7	319	1.2e−7	598	1.1e−7	1157
$1/2^3$	3.8687	1.5e−7	113	1.2e−7	197	1.1e−7	362	1.1e−7	691
$1/2^4$	3.8689	1.5e−7	79	1.3e−7	141	1.1e−7	263	1.1e−7	505
$1/2^5$	3.8689	1.3e−7	74	1.1e−7	122	1.1e−7	218	1.1e−7	409
$1/2^6$	3.8689	1.3e−7	73	1.2e−7	120	1.1e−7	214	1.1e−7	402
$1/2^7$	3.8689	1.3e−7	73	1.1e−7	120	1.1e−7	213	1.1e−7	401

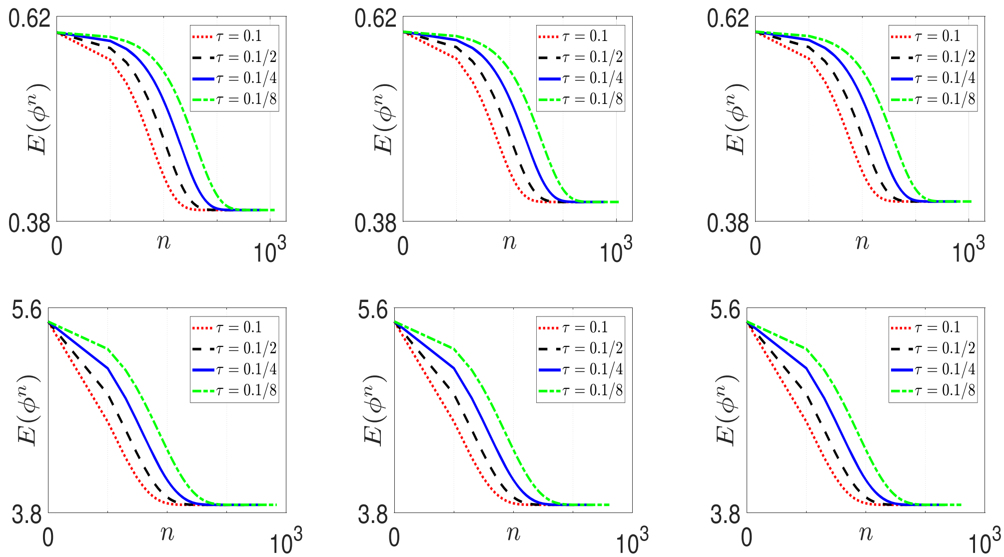

FIG. 1. Evolution of the total energy $E(\phi^n)$ for $\beta = -5$ (upper) and $\beta = 100$ (lower) computed by the GFSI method with different time steps τ and mesh sizes $h = 1/8, 1/32, 1/128$ (from left to right) in Example 1.

We consider various time steps $\tau = 0.1/2^k$ ($k = 0, 1, 2, 3$) and mesh sizes $h = 1/2^\ell$ ($\ell = 2, \ldots, 7$).

Table 1 presents the total energy $E(\phi_g)$, residual r_{ϕ_g} of the converged ground state ϕ_g, computed via GFSI algorithm with various time steps τ and mesh sizes h. The corresponding iteration numbers 'Iter' the GFSI algorithm needed to converge are also listed there. In addition, Fig. 1 illustrates the evolution of the total energy $E(\phi^n)$ for $h = 1/8, 1/32, 1/128$ and different τ. Figure 2 shows the contour plots of the converged density function $|\phi_g|^2$.

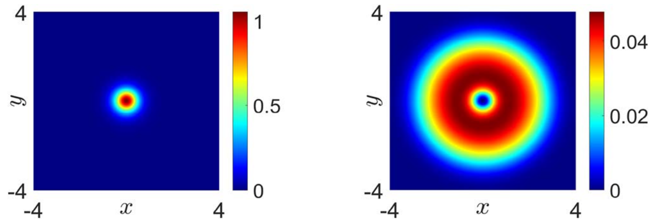

FIG. 2. Contour plots of the converged density function $|\phi_g|^2$ for $\beta = -5$ (left) and $\beta = 100$ (right) in Example 1.

TABLE 2 *Total energy $E(\phi_g)$, residual r_{ϕ_g} of the converged ground state ϕ_g and iterations numbers 'Iter' the GFSI method needed to converged for different β in Example 2*

		$\tau = 1$		$\tau = 0.8$		$\tau = 0.6$		$\tau = 0.4$		$\tau = 0.2$	
β	$E(\phi_g)$	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter	r_{ϕ_g}	Iter
2000	16.8800	2.4e−6	89	1.6e−6	82	1.5e−6	78	9.8e−7	83	5.8e−7	94
3300	21.6553	**2.8e−6**	**116**	2.2e−6	106	1.8e−6	92	1.2e−6	97	7.2e−7	108
4900	26.3702	**3.5e−6**	**142**	**2.9e−6**	**129**	2.0e−6	113	1.6e−6	109	8.7e−7	121
16000	47.6022	**6.9e−6**	**258**	**5.3e−6**	**235**	**4.0e−6**	**204**	2.5e−6	163	1.4e−6	175
50000	84.1210	**1.2e-5**	**458**	**9.9e−6**	**415**	**7.2e−6**	**360**	**4.9e−6**	**285**	2.5e−6	256

From these figures and Table 1 we can see the following: (*i*) For both focusing and defocusing cases the energy diminishing property is verified for the GFSI algorithm with time step $\tau \le \tau_0$ (in this example $\tau_0 = 0.1$). In addition, τ_0 does not depend on the mesh sizes h (cf. Fig. 1). These numerical observations are consistent with the theoretical results proven in Theorem 1. (*ii*) The first-order necessary condition for optimality is satisfied for all tested cases, i.e., $r_{\phi_g} \le 2\times10^{-7}$. Moreover, the GFSI algorithm takes more iterations to converge for a smaller time step τ, while it is not sensitive to the mesh size h. (*iii*) The ground states exist for the focusing case, i.e., $\beta < 0$, and admit a Gaussian profile (cf. Fig. 2). The radius of the Gaussian gets smaller when β becomes smaller.

EXAMPLE 2. Here, we test whether the maximum time step τ_0 that ensures energy dissipation in the GFSI algorithm depends on the parameter β. To this end, we let $\omega = 0$, $L = 16$ and choose the mesh size and initial data for GFSI algorithm, respectively, as $h = 1/16$ and $\phi_0 = e^{-|\mathbf{x}|^2/2}/\sqrt{\pi}$.

We consider a sequence of larger repulsive constants β ranging from 1000 to 50000 and test various time steps $\tau = k/10$ ($k = 2, 4, 6, 8, 10$).

Table 2 lists the total energy $E(\phi_g)$, residual r_{ϕ_g} of the converged ground state ϕ_g and iterations numbers 'Iter' the GFSI method needed to converged for $\beta = 2000$, 3300, 4900, 16000 and 50000 with various time steps τ. While Fig. 3 shows the evolution of those total energy $E(\phi^n)$ for $\beta = 4900$, 16000 and 50000. For each row in Table 2, the bold data indicate that with the corresponding time step τ in that column the GFSI algorithm does not hold the energy-diminishing property. From Table 2 and Fig. 3 we can clearly see that the GFSI algorithm holds energy-diminishing property only when time steps $\tau \le \tau_0$. And such τ_0 decreases as β increases, which is consistent with the theoretical result shown in Theorem 1.

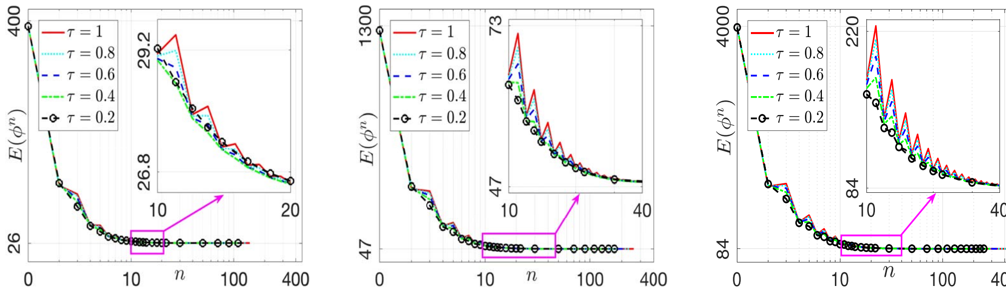

FIG. 3. Evolution of the total energy $E(\phi^n)$ computed by the GFSI algorithm with different time steps τ for $\beta = 4900$, 16000 and 50000 (from left to right) in Example 2.

5. Conclusion

In this work we conducted a thorough study of the normalized GFSI, one of the most classical and popular numerical methods for computing the ground state of rotating BECs. We rigorously proved its energy dissipation and global convergence in both focusing and defocusing cases with and without rotation. These findings address a significant gap in the literature and establish a solid theoretical foundation for the GFSI method.

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Appendix. Proof of Lemma 1

Proof. We start with the proof of (i). By (2.4),

$$(Vu,u)_{L^2} = \int_\Omega V(\mathbf{x})|u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \ge (1+K)\frac{\omega^2}{2}\int_\Omega (x^2+y^2)|u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}. \tag{A.1}$$

In addition, by Young's inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned}\left|(\omega L_z u,u)_{L^2}\right| &\le \int_\Omega |\omega||u||(y\partial_x - x\partial_y)u|\,\mathrm{d}\mathbf{x} \le \int_\Omega \left[\frac{\varepsilon\omega^2}{2}\left(x^2+y^2\right)|u|^2 + \frac{1}{2\varepsilon}\left(|\partial_x u|^2+|\partial_y u|^2\right)\right]\mathrm{d}\mathbf{x}\\ &\le \frac{\varepsilon\,\omega^2}{2}\int_\Omega\left(x^2+y^2\right)|u(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \frac{1}{2\varepsilon}\|u\|^2_{H^1_0}.\end{aligned}$$

Taking $\varepsilon = 1+K$ in the above and using (A.1) we obtain

$$\left|(\omega L_z u,u)_{L^2}\right| \le (Vu,u)_{L^2} + \frac{1}{2(1+K)}\|u\|^2_{H^1_0}, \tag{A.2}$$

which implies

$$\langle H_0 u,u\rangle = \frac{1}{2}\|u\|^2_{H^1_0} + (Vu,u)_{L^2} + (\omega L_z u,u)_{L^2} \ge \frac{1}{2}\|u\|^2_{H^1_0} - \frac{1}{2(1+K)}\|u\|^2_{H^1_0} = C_0\|u\|^2_{H^1_0}.$$

Hence, inequality (2.8) is proved.

Next, we prove (2.9). Since $|V(\mathbf{x})| \le V_\infty := \|V\|_{L^\infty}$ for $\mathbf{x}\in\Omega$, by the Poincaré inequality, we have

$$\left|(Vu,v)_{L^2}\right| \le V_\infty\|u\|_{L^2}\|v\|_{L^2} \le C_p V_\infty\|u\|_{H^1_0}\|v\|_{H^1_0}, \tag{A.3}$$

with C_p given by

$$C_p = C_p(\Omega) = \sup_{0\ne\phi\in H^1_0(\Omega)} \|\phi\|^2_{L^2}/\|\nabla\phi\|^2_{L^2}. \tag{A.4}$$

By Cauchy's inequality and (2.4) we have

$$\begin{aligned}\left|(\omega L_z u,v)_{L^2}\right| &\le |\omega|\int_\Omega |(y\partial_x - x\partial_y)u(\mathbf{x})||v(\mathbf{x})|\,\mathrm{d}\mathbf{x}\\ &\le |\omega|\int_\Omega\Big(|y||v(\mathbf{x})||\partial_x u(\mathbf{x})| + |x||v(\mathbf{x})||\partial_y u(\mathbf{x})|\Big)\,\mathrm{d}\mathbf{x}\\ &\le |\omega|\|yv\|_{L^2}\|\partial_x u\|_{L^2} + |\omega|\|xv\|_{L^2}\|\partial_y u\|_{L^2}\\ &\le |\omega|\sqrt{\|yv\|^2_{L^2}+\|xv\|^2_{L^2}}\sqrt{\|\partial_x u\|^2_{L^2}+\|\partial_y u\|^2_{L^2}}\\ &\le \sqrt{2}\left(\int_\Omega \frac{\omega^2}{2}|\mathbf{x}|^2|v(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\right)^{\frac{1}{2}}\|u\|_{H^1_0}\\ &\le \sqrt{2V_\infty}\|v\|_{L^2}\|u\|_{H^1_0} \le \sqrt{2C_pV_\infty}\|u\|_{H^1_0}\|v\|_{H^1_0}.\end{aligned} \tag{A.5}$$

Recalling (2.6), using (A.3) and (A.5), by Holder's inequality, Gagliardo–Nirenberg inequality and the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, recalling $\|\phi\|_{L^2} = 1$, we obtain the continuity of $\langle H_\phi \cdot, \cdot\rangle$:

$$\begin{aligned}
\left|\langle H_\phi u, v\rangle\right| &= \left|\frac{1}{2}\left(\nabla u, \nabla v\right)_{L^2} + \left((-\omega L_z + V + \beta|\phi|^2)u, v\right)_{L^2}\right| \\
&\le \frac{1}{2}\|u\|_{H_0^1}\|v\|_{H_0^1} + \left|(\omega L_z u, v)_{L^2}\right| + \left|(Vu, v)_{L^2}\right| + |\beta|\|\phi\|_{L^4}^2\|u\|_{L^4}\|v\|_{L^4} \\
&\le \left(\frac{1}{2} + \sqrt{2C_p V_\infty} + C_p V_\infty + C|\beta|\|\phi\|_{L^2}^{2-d/2}\|\phi\|_{H_0^1}^{d/2}\right)\|u\|_{H_0^1}\|v\|_{H_0^1} \\
&= C\left(1 + |\beta|\|\phi\|_{H_0^1}^{d/2}\right)\|u\|_{H_0^1}\|v\|_{H_0^1} = C_\phi\|u\|_{H_0^1}\|v\|_{H_0^1},
\end{aligned}$$

where C depends on Ω, d and $\|V\|_{L^\infty}$.

To prove (*ii*), by (2.9), Cauchy's inequality and Poincaré inequality, we get

$$\begin{aligned}
\left|\mathscr{A}_\phi^a(u, v)\right| &= \left|(u, v)_{L^2} + a\langle H_\phi u, v\rangle\right| \\
&\le \|u\|_{L^2}\|v\|_{L^2} + aC_\phi\|u\|_{H_0^1}\|v\|_{H_0^1} \\
&\le \left(C_p + aC_\phi\right)\|u\|_{H_0^1}\|v\|_{H_0^1}.
\end{aligned}$$

Finally, we prove (*iii*). For the case $\beta > 0$, we have, by (2.8),

$$\mathscr{A}_\phi^a(u, u) = \|u\|_{L^2}^2 + a\langle H_\phi u, u\rangle \ge a\langle H_0 u, u\rangle \ge aC_0\|u\|_{H_0^1}^2.$$

Then, we consider the case $\beta < 0$. By (2.8) and Hölder's inequality we derive

$$\mathscr{A}_\phi^a(u, u) = \|u\|_{L^2}^2 + a\langle H_0 u, u\rangle + a\beta\int_\Omega |\phi|^2|u|^2\,\mathrm{d}\mathbf{x} \ge \|u\|_{L^2}^2 + aC_0\|u\|_{H_0^1}^2 + a\beta\|\phi\|_{L^4}^2\|u\|_{L^4}^2. \tag{A.6}$$

By the Gagliardo–Nirenberg inequality and Young's inequality, recalling $\|\phi\|_{L^2} = 1$, we have

$$\begin{aligned}
|\beta|\|\phi\|_{L^4}^2\|u\|_{L^4}^2 &\le C|\beta|\|\phi\|_{L^2}^{2-d/2}\|\phi\|_{H_0^1}^{d/2}\|u\|_{L^2}^{2-d/2}\|u\|_{H_0^1}^{d/2} \\
&\le C|\beta|\|\phi\|_{H_0^1}^{d/2}\left(\varepsilon^{-d/(4-d)}\|u\|_{L^2}^2 + \varepsilon\|u\|_{H_0^1}^2\right) \\
&= C|\beta|\|\phi\|_{H_0^1}^{d/2}\left(2C|\beta|\|\phi\|_{H_0^1}^{d/2}/C_0\right)^{\frac{d}{4-d}}\|u\|_{L^2}^2 + \frac{C_0}{2}\|u\|_{H_0^1}^2 \\
&=: \frac{1}{a_\phi}\|u\|_{L^2}^2 + \frac{C_0}{2}\|u\|_{H_0^1}^2,
\end{aligned} \tag{A.7}$$

where we have chosen $\varepsilon = C_0/\left(2C|\beta|\|\phi\|_{H_0^1}^{d/2}\right)$ and $a_\phi = C/\left(|\beta|^2\|\phi\|_{H_0^1}^{d}\right)^{\frac{2}{4-d}}$. Plugging (A.7) into (A.6), we have, when $0 < a \le a_\phi$,

$$\mathscr{A}_\phi^a(u, u) \ge \left(1 - \frac{a}{a_\phi}\right)\|u\|_{L^2}^2 + \frac{aC_0}{2}\|u\|_{H_0^1}^2 \ge \frac{aC_0}{2}\|u\|_{H_0^1}^2,$$

which completes the proof. □